

Further results on constructions of generalized bent Boolean functions

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Dear editor,

Boolean bent functions were introduced by Rothaus in 1976 as an interesting combinatorial object with the important property of having optimal nonlinearity [1]. Since bent functions have many applications in sequence design, cryptography and algebraic coding, they have been extensively studied during the last thirty years [2, 3]. Over the past decades, based on bent functions, several constructions of highly nonlinear balanced functions were presented [4, 5].

In recent years several researchers have proposed generalizations of Boolean functions [6–9] and studied the effect of the Walsh-Hadamard transform on these classes. In [6], Schmidt presented the connection between words in multi-code code-division multiple access (MC-CDMA) systems and generalized bent functions from \mathbb{Z}_2^n to \mathbb{Z}_4 , and considered functions from \mathbb{Z}_2^n to \mathbb{Z}_q from the viewpoint of cyclic codes over rings. Later, Solé and Tokareva [7] called these functions from \mathbb{Z}_2^n to \mathbb{Z}_q generalized Boolean functions and presented the direct links between Boolean bent functions and generalized bent functions. More recently, Stănică et al. [9] investigated the properties of generalized bent functions and presented several constructions of such generalized bent functions

for both n even and n odd. They characterized a class of generalized bent functions symmetric with respect to two variables and generalized bent functions defined on \mathbb{Z}_2^n in \mathbb{Z}_8 . However, is there a technique that provides generalized bent functions symmetric with respect to m variables, where m is even? Additionally, in [9, Example 20, 21] the authors provided an explicit construction only for the even case. These give us a motivation to identify those generalized bent functions.

Let us denote the set of integers, real numbers and complex numbers by \mathbb{Z}, \mathbb{R} and \mathbb{C} , respectively and let the ring of integers modulo r be denoted by \mathbb{Z}_r . We denote the addition over \mathbb{Z}, \mathbb{R} and \mathbb{C} by ‘+’. Moreover, addition modulo q ($\neq 2$) is also denoted by ‘+’ and it is understood from the context. Let \mathbb{Z}_2^n be the n -dimensional vector space over \mathbb{Z}_2 . We denote the addition over \mathbb{Z}_2^n and \mathbb{Z}_2 by ‘ \oplus ’. Letting $\omega = (\omega_1, \dots, \omega_n)$ and $x = (x_1, \dots, x_n) \in \mathbb{Z}_2^n$, we define the inner (or scalar) product by $\omega \cdot x = \omega_1 x_1 \oplus \dots \oplus \omega_n x_n$. If $z = a + bi \in \mathbb{C}, a, b \in \mathbb{R}$, then $|z| = \sqrt{a^2 + b^2}$ denotes the absolute value of z , where $i^2 = -1$. We denote the vectors $(0, 0, \dots, 0) \in \mathbb{Z}_2^n$ by $\mathbf{0}_n$.

A function from \mathbb{Z}_2^n to \mathbb{Z}_q ($q \geq 2$ a positive integer) is called a *generalized Boolean function* in n variables [7]. Let \mathcal{GB}_n^q be the set of all n -varia-

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generalized Boolean functions from \mathbb{Z}_2^n to \mathbb{Z}_q . If $q = 2$, we obtain the classical Boolean functions in n variables, whose set will be denoted by \mathcal{B}_n . The *Hamming weight* $\text{wt}(\mathbf{u})$ of a vector $\mathbf{u} \in \mathbb{Z}_2^n$ is the weight (number of 1's) of the binary string.

The (*generalized*) *Walsh-Hadamard transform* of $f \in \mathcal{GB}_n^q$ is the complex valued function over \mathbb{Z}_2^n which is defined by $\mathcal{H}_f(\boldsymbol{\omega}) = \sum_{\mathbf{x} \in \mathbb{Z}_2^n} \zeta^{f(\mathbf{x})} (-1)^{\boldsymbol{\omega} \cdot \mathbf{x}}$ where $\zeta (= e^{2\pi i/q})$ is the complex q -primitive root of unity. When $q = 2$, we obtain the Walsh transform of $f \in \mathcal{B}_n$, which will be denoted by \mathcal{W}_f .

A generalized Boolean function $f \in \mathcal{GB}_n^q$ is called *generalized bent* (or *gbent*, for short) if and only if $|\mathcal{H}_f(\boldsymbol{\omega})| = 2^{n/2}$ for all $\boldsymbol{\omega} \in \mathbb{Z}_2^n$. Note that when $q = 2$, Boolean bent functions exists only if the number n of variables is even. For $q > 2$, if f is a gbent function in n variables, it does not follow that n must be even. Such functions for $q = 4$ were investigated by Schmidt [6], Solé and Tokareva [7], Stănică, Martinsen, Gangopadhyay, and Singh [9], etc.

The sum $\mathcal{C}_{f,g}(\mathbf{u}) = \sum_{\mathbf{x} \in \mathbb{Z}_2^n} \zeta^{f(\mathbf{x}) - g(\mathbf{x} \oplus \mathbf{u})}$ is the *crosscorrelation* of f and g at $\mathbf{u} \in \mathbb{Z}_2^n$. The *auto-correlation* of $f \in \mathcal{GB}_n^q$ at $\mathbf{u} \in \mathbb{Z}_2^n$ is $\mathcal{C}_{f,f}(\mathbf{u})$ above, which is denoted by $\mathcal{C}_f(\mathbf{u})$.

Lemma 1. Let $f \in \mathcal{GB}_n^q$. Then f is a gbent function if and only if

$$\mathcal{C}_f(\mathbf{u}) = \begin{cases} 2^n, & \text{if } \mathbf{u} = \mathbf{0}_n, \\ 0, & \text{if } \mathbf{u} \neq \mathbf{0}_n. \end{cases}$$

By using Lemma 1, we can prove the following theorem.

Theorem 1. Let n be a positive integer and m, q be even positive integers. Let $f \in \mathcal{GB}_n^q$ be gbent. Let $f + \frac{q}{2}g_i \in \mathcal{GB}_n^q$ be gbent, where $i = 0, 1$. Let $\mathbf{y} = (\mathbf{y}', \mathbf{y}'')$, $\mathbf{y}' = (y_1, y_2, \dots, y_{m/2})$, $\mathbf{y}'' = (y_{m/2+1}, y_{m/2+2} \dots y_m)$ and $\vartheta(\mathbf{y}) = \mathbf{y}' \cdot \mathbf{y}''$. Let $\mathbf{c} \in \mathbb{Z}_2^m$ and $\text{wt}(\mathbf{c})$ be even. Then the function $h \in \mathcal{GB}_n^q$, defined by

$$h(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) + \frac{q}{2}(\mathbf{c} \cdot \mathbf{y})g_{\mathbf{c} \cdot \mathbf{y}}(\mathbf{x}) + \frac{q}{2}\vartheta(\mathbf{y}) \quad (1)$$

is a gbent function in $n + m$ variables.

In Table 1, we compare our approach to other methods [9, 10] in terms of the form of gbent functions.

In what follows, we first provide some notations.

If $f \in \mathcal{B}_n$ is bent, then the dual function \tilde{f} of f , defined on \mathbb{Z}_2^n by $\mathcal{W}_f(\boldsymbol{\omega}) = 2^{n/2}(-1)^{\tilde{f}(\boldsymbol{\omega})}$ is also bent and it is known that $\tilde{\tilde{f}} = f$.

Lemma 2. For every $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_2^n$ and for every bent function f , the dual of the function $f(\mathbf{x} \oplus \mathbf{b}) \oplus \mathbf{a} \cdot \mathbf{x}$ equals $\tilde{f}(\mathbf{x} \oplus \mathbf{a}) \oplus \mathbf{b} \cdot (\mathbf{x} \oplus \mathbf{a})$.

The original Maiorana-McFarland's (*M-M*) class of bent functions is the set of all the (bent) Boolean functions on $\mathbb{Z}_2^{2n} = \{(\mathbf{x}, \mathbf{y}) | \mathbf{x}, \mathbf{y} \in \mathbb{Z}_2^n\}$ of the form

$$f(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \phi(\mathbf{y}) \oplus g(\mathbf{y}), \quad (2)$$

where ϕ is any permutation of \mathbb{Z}_2^n and $g \in \mathcal{B}_n$.

Let $f \in \mathcal{B}_n$. If there exists an even integer $0 \leq r \leq n$, such that $\|\{\boldsymbol{\omega} | \mathcal{W}_f(\boldsymbol{\omega}) \neq 0, \boldsymbol{\omega} \in \mathbb{F}_2^n\}\| = 2^r$, where $\|\cdot\|$ denotes the size (cardinality) of a set, and $(\mathcal{W}_f(\boldsymbol{\omega}))^2$ equals 2^{2n-r} or 0, for every $\boldsymbol{\omega} \in \mathbb{F}_2^n$, then f is called an *r-order plateaued* function in n variables. If f is a $2\lceil(n-2)/2\rceil$ -order plateaued function in n variables, then f is also called a *semibent* function.

Let $f \in \mathcal{GB}_n^8$ be as

$$f(\mathbf{x}) = v_0(\mathbf{x}) + v_1(\mathbf{x}) \cdot 2 + v_2(\mathbf{x}) \cdot 2^2, \quad (3)$$

where $v_i(\mathbf{x}) \in \mathcal{B}_n, i = 0, 1, 2$.

In [9, Theorem 19], Stănică et al. presented a sufficient and necessary condition for a function f as in (3) to be gbent.

Theorem 2 ([9]). Let $f \in \mathcal{GB}_n^8$ be as in (3). The following are true:

(i) If n is even, then f is gbent if and only if $v_2, v_0 \oplus v_2, v_1 \oplus v_2, v_0 \oplus v_1 \oplus v_2$ are all bent, and $\mathcal{W}_{v_0 \oplus v_2}(\mathbf{u})\mathcal{W}_{v_1 \oplus v_2}(\mathbf{u}) = \mathcal{W}_{v_2}(\mathbf{u})\mathcal{W}_{v_0 \oplus v_1 \oplus v_2}(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{Z}_2^n$;

(ii) If n is odd, then f is gbent if and only if $v_2, v_0 \oplus v_2, v_1 \oplus v_2, v_0 \oplus v_1 \oplus v_2$ are all semibent, and $\mathcal{W}_{v_0 \oplus v_2}(\mathbf{u}) = \mathcal{W}_{v_2}(\mathbf{u}) = 0$ and $|\mathcal{W}_{v_1 \oplus v_2}(\mathbf{u})| = |\mathcal{W}_{v_0 \oplus v_1 \oplus v_2}(\mathbf{u})| = 2^{\frac{n+1}{2}}$; or, $|\mathcal{W}_{v_0 \oplus v_2}(\mathbf{u})| = |\mathcal{W}_{v_2}(\mathbf{u})| = 2^{\frac{n+1}{2}}$ and $\mathcal{W}_{v_1 \oplus v_2}(\mathbf{u}) = \mathcal{W}_{v_0 \oplus v_1 \oplus v_2}(\mathbf{u}) = 0$, for all $\mathbf{u} \in \mathbb{Z}_2^n$.

From the above theorem, we know that the sufficient conditions that a function f as in (3) is gbent are abstract. Hence, we provide some sufficient conditions for a function f as in (3) to be gbent.

Theorem 3. Let n be an even integer, $v_0, v_1, v_2 \in \mathcal{B}_n$ and $f \in \mathcal{GB}_n^8$ be as in (3). The following v_0, v_1, v_2 satisfy the sufficient conditions of Theorem 2 for the even case.

(i) Let v_0, v_1, v_2 be bent functions and $v_2, v_0 \oplus v_2, v_1 \oplus v_2, v_0 \oplus v_1 \oplus v_2$ be all bent, and $(\tilde{v}_0 \oplus v_2)(\mathbf{x}) = \tilde{v}_0(\mathbf{x}) \oplus \tilde{v}_2(\mathbf{x})$, $(\tilde{v}_1 \oplus v_2)(\mathbf{x}) = \tilde{v}_1(\mathbf{x}) \oplus \tilde{v}_2(\mathbf{x})$, $(v_0 \oplus v_1 \oplus v_2)(\mathbf{x}) = \tilde{v}_0(\mathbf{x}) \oplus \tilde{v}_1(\mathbf{x}) \oplus \tilde{v}_2(\mathbf{x})$.

(ii) Let $v_2 \in \mathcal{B}_n$ be a bent function, $v_0 = v_1$ and $v_0 \oplus v_2$ be bent.

(iii) Let $v_0(\mathbf{x}) = \mathbf{a}_0 \cdot \mathbf{x}$ and $v_1(\mathbf{x}) = \mathbf{a}_1 \cdot \mathbf{x}$ respectively, be two linear functions. Let $v_2 \in \mathcal{B}_n$ be a bent function, and $\tilde{v}_2(\mathbf{x}) \oplus \tilde{v}_2(\mathbf{x} \oplus \mathbf{a}_0) \oplus \tilde{v}_2(\mathbf{x} \oplus \mathbf{a}_1) \oplus \tilde{v}_2(\mathbf{x} \oplus \mathbf{a}_0 \oplus \mathbf{a}_1) = 0$.

Table 1 Form of gbent functions comparison

Number of variables	q	From	Resource
$n + 2$	2	$h(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) \oplus (y_1 \oplus y_2)g(\mathbf{x}) \oplus y_1 y_2$	Ref. [10]
$n + 2$	Even integer	$h(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) + (y_1 \oplus y_2)g(\mathbf{x}) + \frac{q}{2}y_1 y_2$	Ref. [9]
$n + m$	Even integer	$h(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) + \frac{q}{2}(\mathbf{c} \cdot \mathbf{y})g_{\mathbf{c} \cdot \mathbf{y}}(\mathbf{x}) + \frac{q}{2}\vartheta(\mathbf{y})$	New

(iv) Let $v_0(\mathbf{x}) = \mathbf{a}_0 \cdot \mathbf{x}$, be a linear function. Let $v_2 \in \mathcal{B}_n$ be a bent function, $v_1 \oplus v_2$ be bent and $\widetilde{v}_2(\mathbf{x}) \oplus \widetilde{v}_2(\mathbf{x} \oplus \mathbf{a}_0) \oplus (v_1 \oplus v_2)(\mathbf{x}) \oplus (v_1 \oplus v_2)(\mathbf{x} \oplus \mathbf{a}_0) = 0$.

We now discuss the case when n is odd. Let n be a positive odd integer and $g_1, g_2 \in \mathcal{B}_n$. We say that g_1 and g_2 are *complementary semibent functions* in n variables if they are semibent (that is, $(n - 1)$ -order plateaued) functions and satisfy the property that $\mathcal{W}_{g_1}(\omega) = 0$ if and only if $\mathcal{W}_{g_2}(\omega) \neq 0$.

Lemma 3. Let n be an even integer and $f \in \mathcal{B}_n$. Then f is bent if and only if the two functions on \mathbb{Z}_2^{n-1} , $f(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n)$ and $f(x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_n)$, are complementary semibent functions on \mathbb{Z}_2^{n-1} , where $j = 1, \dots, n$.

Theorem 4. Let k, n be two integers and $n = 2k - 1$. Let $\varphi = (\varphi_1, \dots, \varphi_k), \phi = (\phi_1, \dots, \phi_k)$ be Boolean maps from \mathbb{Z}_2^k to \mathbb{Z}_2^k such that both ϕ and $\phi \oplus \varphi = (\phi_1 \oplus \varphi_1, \dots, \phi_k \oplus \varphi_k)$ are permutations on \mathbb{Z}_2^k . Set $\Delta_j = \{\phi(\mathbf{y}) | \mathbf{y} \in \mathbb{Z}_2^{j-1} \times \{0\} \times \mathbb{Z}_2^{k-j}\}$, $\mathbf{y}_\epsilon^{(j)} = (y_1, \dots, y_{j-1}, \epsilon, y_{j+1}, \dots, y_k)$, where $\epsilon \in \mathbb{Z}_2, j = 1, 2, \dots, k$. Let $f \in \mathcal{GB}_n^8$ be as in (3), and let $v_0(\mathbf{x}, \mathbf{y}_0^{(j)}) = \mathbf{a}_0 \cdot \mathbf{x} \oplus \varphi(\mathbf{y}_0^{(j)}) \cdot \mathbf{x}$, $v_1(\mathbf{x}) = (\phi(\mathbf{y}_0^{(j)}) \oplus \phi(\mathbf{y}_1^{(j)})) \cdot \mathbf{x} \oplus g(\mathbf{y}_0^{(j)}) \oplus g(\mathbf{y}_1^{(j)})$ and $v_2(\mathbf{x}) = \phi(\mathbf{y}_0^{(j)}) \cdot \mathbf{x} \oplus g(\mathbf{y}_0^{(j)})$, where $\mathbf{a}_0 \in \mathbb{Z}_2^k$. If there exists one positive integer $\varrho (\leq k)$ such that

$$\{(\phi \oplus \varphi)(\mathbf{y}) | \mathbf{y} \in \mathbb{Z}_2^{\varrho-1} \times \{0\} \times \mathbb{Z}_2^{k-\varrho}\} = \Delta_\varrho \quad (4)$$

(if $\mathbf{a}_0 \neq \mathbf{0}_k$ we further require Δ_ϱ to be a linear subspace of \mathbb{Z}_2^k and $\mathbf{a}_0 \in \Delta_\varrho$), then v_0, v_1, v_2 satisfy the conditions of Theorem 2 for the odd case, that is, f is gbent.

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